Invertible coupled KdV and coupled Harry Dym hierarchies

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Abstract

In this paper we discuss the conditions under which the coupled KdV and coupled Harry Dym hierarchies possess inverse (negative) parts. We further investigate the structure of nonlocal parts of tensor invariants of these hierarchies, in particular, the nonlocal terms of vector fields, conserved one-forms, recursion operators, Poisson and symplectic operators. We show that the invertible cKdV hierarchies possess Poisson structures that are at most weakly nonlocal while coupled Harry Dym hierarchies have Poisson structures with nonlocalities of the third order.

Keywords and phrases: Energy dependent Schrödinger spectral problem, invertible cKdV and cHD hierarchies, recursion operator, nonlocal structures

1 Introduction

The energy-dependent Schrödinger spectral problem has been introduced by Jaulent and Miodek in [1] in the two-field case. It has been generalized to an arbitrary number of components by Martínez Alonso [2] who also presented its multi-Hamiltonian structure and gave the problem its present name. Antonowicz and Fordy have further investigated this problem in a series of papers ([3],[4],[5],[6],[7]). They demonstrated that this spectral problem leads to two families of coupled (multicomponent) soliton hierarchies: the coupled KdV (cKdV) and the coupled Harry-Dym (cHD) hierarchies. In their approach one simultaneously obtains the evolution equations of the hierarchy together with the set of independent closed one-forms (variational derivatives of Hamiltonians), which can be obtained with the help of a recursion relation, solvable under additional conditions. The specification of these conditions fixes the type of the hierarchy (either cKdV or cHD).

In this paper we complete their work by finding the conditions under which both types of hierarchies have also negative parts (i.e., when the recursion operator is explicitly invertible) and present a first few flows of these negative hierarchies. Further, using the results from [8] and [9], we investigate the structure of nonlocal parts of all Hamiltonian structures associated with both types of hierarchies: we show that all Hamiltonian and symplectic structures of the cKdV-type hierarchies are at most weakly nonlocal while the cHD-type hierarchies have Hamiltonian and symplectic structures that are nonlocal up to third order, i.e., they have terms up to ∂^{-3} in their nonlocal parts. We also present compact formulas for the highest nonlocalities of all these quantities.

2 The spectral problem

We start by recalling the basic facts about coupled (multicomponent) KdV and Harry Dym hierarchies following the papers [4], [5] and [7] (see also [10]). Let us consider the Schrödinger equation

$$\mathcal{L}\Psi \equiv (\varepsilon\partial^2 + u)\Psi = 0 \tag{1}$$

where $\Psi = \Psi(x,t)$ and $u = u(x,t)$ are smooth functions of x and t and where $\partial = \frac{\partial}{\partial x}$, together with the auxiliary linear problem of the form

$$\Psi_t = \left(\tfrac{1}{2}P\partial + Q\right)\Psi. \tag{2}$$

The functions P and Q will be specified below.

The system (1)-(2) is compatible, i.e., has nontrivial solutions for Ψ provided that $(\mathcal{L}\Psi)_t = 0$ which yields

$$\frac{1}{2}\varepsilon(P_{xx} + 4Q_x)\Psi_x + \left(\varepsilon Q_{xx} - uP_x - \frac{1}{2}Pu_x + u_t\right)\Psi = 0.$$

Here and below the subscripts x and t denote the derivatives with respect to x and t respectively. Thus, the equations

$$P_{xx} + 4Q_x = 0, \qquad \varepsilon Q_{xx} - uP_x - \frac{1}{2}Pu_x + u_t = 0$$

constitute a set of sufficient conditions for the existence of a common solution $\Psi(x,t)$ to (1)-(2). The first equation can be replaced by $Q = -\frac{1}{4}P_x$, and then the second one reads

$$u_t = JP \equiv \left(\frac{1}{4}\varepsilon\partial^3 + \frac{1}{2}(u\partial + \partial u)\right)P, \tag{3}$$

where J is a third-order differential operator given above. Assume now that both ε and u are polynomial functions (of the same degree) of a new parameter λ so that

$$u = \sum_{k=0}^{N} u_k(x,t)\lambda^k, \quad \varepsilon = \sum_{k=0}^{N}\varepsilon_k\lambda^k, \tag{4}$$

where ε_k are $N+1$ arbitrary (so far) real constants. Then J is polynomial as well: $J = \sum_{k=0}^{N} J_k\lambda^k$ with

$$J_k = \frac{1}{4}\varepsilon_k\partial^3 + \frac{1}{2}(u_k\partial + \partial u_k), \quad k = 0,\dots,N. \tag{5}$$

Assume further that also P (and thus Q) depends polynomially on λ so that

$$P = \sum_{k=0}^{m} P_k\lambda^{m-k}, \quad m \in \mathbf{N}. \tag{6}$$

It is not the only possibility (for example, in [10] we have considered a rational dependence of P on λ which leads to hierarchies with sources) but here we restrict ourselves to (6). Plugging (4) and (6) into (3) we obtain, for any fixed $m \in \mathbf{N}$, the following $(N+1)$-component system of evolution equations

$$u_{r,t_m} = J_0P_{m-r} + \cdots + J_rP_m \ , \ r = 0,\dots,N \tag{7}$$

(with $P_i = 0$ for $i < 0$) and the following recursion on P_i

$$J_0P_{k-N} + J_1P_{k-N+1} + \cdots + J_NP_k = 0, \quad k = 0,\dots,m-1. \tag{8}$$

Note that the natural parameters N and m are independent; N will be the number of fields in the hierarchies that originate in this scheme while m enumerates the flows within a chosen hierarchy. The recursion (8) starts at $k = 0$ with the equation $J_NP_0 = 0$ and it can be effectively solved either when $u_N = 0$ or when $\varepsilon_N = 0$. In the case of $u_N = 0$ we have $J_N = \frac{1}{4}\varepsilon\partial^3$ which leads to P_0 and thus P_r depending explicitly on x which we do not consider here. We must therefore assume (and will stick

to this throughout the whole article) that $\varepsilon_N = 0$ so that $J_N = \frac{1}{2}(u_N\partial + \partial u_N) = u_N^{1/2}\partial u_N^{1/2}$ which implies that J_N is invertible with the inverse $J_N^{-1} = u_N^{-1/2}\partial^{-1}u_N^{-1/2}$, where ∂^{-1} is a formal inverse of ∂ ($\partial\partial^{-1} = \partial^{-1}\partial = 1$). The recursion (8) allows now to compute P_0, P_1 and so on up to P_{m-1} as differential functions of u with P_m undetermined since P_m does not enter the formulas (8). In the construction given in [7] one can demonstrate that in order to embed (7) into an infinite hierarchy with well-defined and explicitly computable P_m for all $m \in \mathbf{N}$ one must make a reduction of the system (7) to an N-component system by assuming either that u_N is constant (to make the coefficients in the hierarchy as simple as possible, the most convenient choice is to set $u_N = -1$) or that $u_0 = -a^2$ (a is a constant). The first choice leads to the coupled KdV hierarchy, the second to the coupled Harry Dym (cHD) hierarchy. But, in fact, under the additional assumption ($\varepsilon_0 = 0$ in the KdV case and $u_0 = -a^2 = 0$ in the HD case) one can invert the operator J_0 which makes it possible to construct both KdV and HD negative hierarchies even in the multicomponent (coupled) case. The existence of the inverse cKdV hierarchy was mentioned in [7], but never treated in detail. The one-field case of the inverse HD hierarchy was considered in [11].

There is actually a third possibility of obtaining a reversible hierarchy in this scheme: by putting $\varepsilon_N \neq 0$ but $u_N = 0$ and $\varepsilon_0 = 0$ we also arrive at an invertible hierarchy but it can be shown that this hierarchy is just a reparametrization of the invertible coupled Harry Dym hierarchy described below.

3 Some algebraic structures and their nonlocal parts

In order to state our results we need to introduce some algebraic objects and to discuss their nonlocal parts. Following Antonowicz and Fordy, denote by B_0 the Hamiltonian operator

$$B_0 = \begin{pmatrix} -J_1 & -J_2 & \cdots & -J_N \\ -J_2 & & -J_N & \\ \vdots & \cdots & & \\ -J_N & & & \end{pmatrix}$$

and by R the following nonlocal operator of $(1,1)$-type

$$R = \left(\begin{array}{ccc|c} 0 & \dots & 0 & -J_0 J_N^{-1} \\ \hline & & & -J_1 J_N^{-1} \\ & I_{N-1} & & \vdots \\ & & & -J_{N-1} J_N^{-1} \end{array}\right) \tag{9}$$

where, as in the previous section,

$$J_i = \frac{1}{4}\varepsilon_i\partial^3 + \frac{1}{2}(u_i\partial + \partial u_i), \quad i = 0, \dots, N \tag{10}$$

with $u_i = u_i(x,t)$, $i = 0, \dots, N$, and with $\varepsilon_N = 0$ so that $J_N = \frac{1}{2}(u_N\partial + \partial u_N) = u_N^{1/2}\partial u_N^{1/2}$ is invertible. Here and below I_{N-1} stands for the $(N-1)\times(N-1)$ unit matrix. Note also that (9) implies that the operator $R^\dagger$ adjoint to R has the form

$$R^\dagger = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{N-1} & \\ 0 & & & \\ \hline -J_N^{-1}J_0 & -J_N^{-1}J_1 & \cdots & -J_N^{-1}J_{N-1} \end{array}\right) \tag{11}$$

(since $J_i^\dagger = -J_i$ for all i). It can be proved that the operator R is a hereditary recursion operator [12] meaning that its Nijenhuis torsion vanishes. Let us now assume that the operator J_0 is also invertible (this can be achieved either by putting $\varepsilon_0 = 0$ or by putting $u_0 = 0$, as discussed below). Then R is invertible with

$$R^{-1} = \left(\begin{array}{c|ccc} -J_1 J_0^{-1} & & & \\ \vdots & & I_{N-1} & \\ -J_{N-1}J_0^{-1} & & & \\ \hline -J_N J_0^{-1} & 0 & \cdots & 0 \end{array}\right)$$

which implies

$$\left(R^{-1}\right)^{\dagger}=\left(\begin{array}{ccc|c} -J_0^{-1}J_1 & \cdots & -J_0^{-1}J_{N-1} & -J_0^{-1}J_N \\ \hline & & & 0 \\ & I_{N-1} & & \vdots \\ & & & 0 \end{array}\right).$$

Now, since R is invertible, and due to the fact that R is hereditary, the infinite family of operators

$$B_s = R^s B_0, \quad s \in \mathbf{Z} \tag{12}$$

is a family of compatible Poisson operators. The operators B_s are purely local for $s = 0, \ldots, N$; explicitly, they have the form

$$B_s=\left(\begin{array}{cccc|cccc} & & & J_0 & & & & \\ & & J_0 & J_1 & & & 0 & \\ & \cdots & & \vdots & & & & \\ J_0 & J_1 & \cdots & J_{s-1} & & & & \\ \hline & & & & -J_{s+1} & \cdots & -J_{N-1} & -J_N \\ & & & & \vdots & & \cdots & \\ & 0 & & & -J_{N-1} & -J_N & & \\ & & & & -J_N & & & \end{array}\right), \quad s=0,\ldots,N \tag{13}$$

so that B_0 is as above and

$$B_N=\begin{pmatrix} & & & J_0 \\ & & J_0 & J_1 \\ & \cdots & & \vdots \\ J_0 & J_1 & \ldots & J_{N-1} \end{pmatrix}.$$

As we will show below, all the other B_s are nonlocal.

Since B_0 is invertible and Poisson, its inverse $\Omega_0 = B_0^{-1}$ is a closed two-form. We can therefore also define a family of two-forms

$$\Omega_s = \left(R^{\dagger}\right)^s \Omega_0,\ s \in \mathbf{Z}. \tag{14}$$

Moreover, according to the theorem given below, $\Omega_s^{-1} = B_{-s}$ so that all Ω_s constitute a family of closed two-forms.

Theorem 1 *For the families B_s and Ω_s defined above we have*

$$B_s\Omega_{-s} = I_N \text{ for } s \in \mathbf{Z}.$$

Proof. Since $B_s = R^s B_0$ we have $R^s = B_s\Omega_0$ so that $\left(R^{\dagger}\right)^s = \Omega_0 B_s$ (note that $\Omega_0^{\dagger} = -\Omega_0$ and $B_s^{\dagger} = -B_s$) and thus $\Omega_s = \left(R^{\dagger}\right)^s \Omega_0 = \Omega_0 B_s \Omega_0$. We have then

$$B_s\Omega_{-s} = R^s B_0 \Omega_0 B_{-s}\Omega_0 = R^s B_{-s}\Omega_0 = R^s R^{-s} B_0\Omega_0 = I_N$$

■

Let us now choose two one-forms γ_0 and γ_{-1} (they will be "starting points" of our hierarchies later on) so that

$$\gamma_0 = (0, \ldots, 0, \varphi)^T \text{ with } \varphi \in \mathrm{Ker} J_N, \quad \gamma_{-1} = (\psi, 0, \ldots, 0)^T \text{ with } \psi \in \mathrm{Ker} J_0. \tag{15}$$

We have then

$$\gamma_0 \in \mathrm{Ker}\left((R^{\dagger})^{-1}\right), \ \ \gamma_{-1} \in \mathrm{Ker}(R^{\dagger})$$

so that

$$\left(R^{\dagger}\right)^{-1}\gamma_0 = 0, \quad R^{\dagger}\gamma_{-1} = 0. \tag{16}$$

Moreover, it is easy to check that γ_0 belongs to the kernel of each of the operators $B_0, \ldots, B_{N-1}$

$$\gamma_0 \in \mathrm{Ker} B_s \quad \text{for all } s = 0, \ldots, N-1. \tag{17}$$

We define now two families of one-forms

$$\gamma_s = \left(R^{\dagger}\right)^s \gamma_0, \qquad \gamma_{-s} = \left(\left(R^{-1}\right)^{\dagger}\right)^{s-1} \gamma_{-1}, \quad s = 1, 2, \ldots \tag{18}$$

and it follows from (16) and (18) that for any two non-negative s and p we have

$$\left(R^{\dagger}\right)^s \gamma_{-p} = \begin{cases} \gamma_{-p+s} \text{ for } s < p \\ 0 \text{ for } s \geq p \end{cases} \quad \text{and} \quad \left(\left(R^{-1}\right)^{\dagger}\right)^s \gamma_p = \begin{cases} \gamma_{p-s} \text{ for } s \leq p \\ 0 \text{ for } s > p \end{cases} \tag{19}$$

We can now show exactly which γ_i belong to $\mathrm{Ker} B_s$ for any given $s \in Z$.

Lemma 2 *From the property (17) it follows that*

$$\begin{gathered} \gamma_{-s}, \ldots, \gamma_{-s+N-1} \in Ker(B_s) \textit{ for } 0 \leq s \leq N, \\ \gamma_{-N-s}, \ldots, \gamma_{-1} \in Ker(B_{N+s}) \textit{ for } s > 0, \\ \gamma_0, \ldots, \gamma_{s+N-1} \in Ker(B_{-s}) \textit{ for } s > 0. \end{gathered} \tag{20}$$

Proof. The condition (17) means that $B_s\gamma_0 = 0$ for $s = 0, \ldots, N-1$. But $B_s\gamma_0 = R^s B_0 \gamma_0 = B_0 \left(R^{\dagger}\right)^s \gamma_0$ $= B_0\gamma_s$ so that

$$\gamma_0, \ldots, \gamma_{N-1} \in Ker(B_0). \tag{21}$$

Further, if $0 \leq s \leq N$ and if $p \geq 0$ then $B_s\gamma_p = R^s B_0 \gamma_p = B_0 \left(R^{\dagger}\right)^s \gamma_p = B_0 \gamma_{p+s} = 0$ as soon as $p + s \leq N - 1$ (by (21)), i.e. as soon as $p \leq N - s - 1$. If we still have $0 \leq s \leq N$ but $p < 0$ then by the same calculation $B_s\gamma_p = B_0 \left(R^{\dagger}\right)^s \gamma_p = 0$ whenever $s \geq -p$ or $p \geq -s$ (due to the first formula in (19)). This proves the first formula in (20). The argument for B_{N+s} is similar. Finally, for $s > 0$, $B_{-s}\gamma_p = B_0 \left(R^{\dagger}\right)^{-s} \gamma_p$ is equal to 0 for $0 \leq -s + p \leq N - 1$ (by (21)) and it is also equal to 0 for $p = 0, \ldots, s-1$ (by the second formula in (19)). This yields the third formula in (20). ■

Thus, the purely local Poisson tensors B_s, $s = 0, \ldots, N$ have N Casimir one-forms γ_i each, while all the other Poisson tensors B_s,(which will be shown to be nonlocal) have $N + s$ Casimir one-forms γ_i each.

In this article we will study the leading nonlocal terms of invariant tensor objects (vector fields, closed one-forms, closed two-forms, Poisson tensors and recursion operators) for the invertible coupled KdV and coupled Harry Dym hierarchies. In order to do this, we have to establish some facts about nonlocal linear differential operators. Throughout the whole article we will deal with linear matrix pseudo-differential operators of the form

$$\Phi = \Phi_{>-n} + \sum_{\alpha=1}^{p} W_\alpha \partial^{-n} \varphi_\alpha^T, \tag{22}$$

where $n \in \mathbf{Z}_+$, W_α and φ_α are some column matrices with entries being some functions of x and where $\Phi_{>-n}$ denotes the part of the operator Φ that contains all the local terms and all the nonlocal terms up to ∂^{-n+1}. We will also call the number n the *order of nonlocality* of the operator Φ. By Φ_{-n} we mean the highest nonlocal term of the operator Φ in (22), i.e.,

$$\Phi_{-n} = \sum_{\alpha=1}^{p} W_\alpha \partial^{-n} \varphi_\alpha^T.$$

We will now state an important theorem that generalizes formula (7) from [9].

Theorem 3 *Consider two linear matrix nonlocal differential operators of the form*

$$\Phi = \Phi_{>-n} + \sum_{\alpha=1}^{p} W_\alpha \partial^{-n} \varphi_\alpha^T \quad \textit{and} \quad \widetilde{\Phi} = \widetilde{\Phi}_{>-m} + \sum_{\alpha=1}^{\widetilde{p}} \widetilde{W}_\alpha \partial^{-m} \widetilde{\varphi}_\alpha^T \tag{23}$$

where m and n are some natural numbers and where $W_\alpha, \varphi_\alpha, \widetilde{W}_\alpha, \widetilde{\varphi}_\alpha$ are some column matrices with entries being some functions of x. Assume that all the products $\varphi_\alpha \widetilde{W}_\beta$ in (23) are not constant. Then, the product $\Phi\ \widetilde{\Phi}$ has nonlocality of order $\max\{m,n\}$ and its highest nonlocal term is in the case of $n > m$ given by

$$\left(\Phi\ \widetilde{\Phi}\right)_{-n} = \sum_{\alpha=1}^{p} W_\alpha \partial^{-n} \left[\widetilde{\Phi}^\dagger (\varphi_\alpha)\right]^T$$

in case of $n = m$ it is given by

$$\left(\Phi\ \widetilde{\Phi}\right)_{-n} = \sum_{\alpha=1}^{p} W_\alpha \partial^{-n} \left[\widetilde{\Phi}^\dagger (\varphi_\alpha)\right]^T + \sum_{\alpha=1}^{\widetilde{p}} \Phi\left(\widetilde{W}_\alpha\right) \partial^{-n} \widetilde{\varphi}_\alpha^T$$

and in the case of $n < m$ it is given by

$$\left(\Phi\ \widetilde{\Phi}\right)_{-m} = \sum_{\alpha=1}^{\widetilde{p}} \Phi\left(\widetilde{W}_\alpha\right) \partial^{-m} \widetilde{\varphi}_\alpha^T.$$

The proof of this theorem consists in a rather straightforward computation based on the following lemma:

Lemma 4 *If f is a non-constant function of x then*

$$\partial^{-n} f \partial^{-m} = \begin{cases} \left(\partial^{-n} f\right) \partial^{-m} + \textit{lower}, \ \ \textit{for } n < m \\ \left(\partial^{-n} f\right) \partial^{-n} + (-1)^n \partial^{-n} \left(\partial^{-n} f\right) + \ \textit{lower}, \ \ \textit{for } n = m \\ (-1)^m \partial^{-n} \left(\partial^{-m} f\right) + \textit{lower}, \textit{ for } n > m \end{cases}$$

where the word "lower" means nonlocal terms of lower order of nonlocality.

One proves this lemma by repeated integration by parts. Now, the repeated use of Theorem 3 leads to

Theorem 5 *Suppose that no $\varphi_\alpha W_\beta$ in the operator Φ given by (22) is constant. Then the s-th power Φ^s ($s \in \mathbf{Z}_+$) of the operator Φ has nonlocality of the same order as Φ and the following formula is valid*

$$(\Phi^s)_{-n} = \sum_{j=0}^{s-1} \sum_{\alpha=1}^{p} \Phi^j (W_\alpha) \partial^{-n} \left[\left(\Phi^\dagger\right)^{s-1-j} (\varphi_\alpha)\right]^T. \tag{24}$$

One proves this theorem by induction. A special case of this theorem (for $n = 1$) can be found in [9], but it includes the erronous coefficients $\binom{s-1}{j}$ that are not present in the correct version of the formula.

4 Invertible coupled KdV hierarchy

We are now in position to define our invertible hierarchies. The invariant one-forms of both hierarchies will be generated by the formulas (18). Let us start with the invertible coupled KdV hierarchy. This hierarchy originates from (7) when we set $\varepsilon_0 = \varepsilon_N = 0$, $u_N = -1$ and use the powers of $R^\dagger$ and $(R^\dagger)^{-1}$ as described above.

Definition 6 *The invertible N-component coupled KdV (invertible cKdV) hierarchy is the family of flows (i.e. systems of evolutionary PDE's)*

$$\begin{aligned} \frac{d}{dt_{-s}} u &= K_{-s} \equiv B_r \gamma_{-r-s}, \ \ s = 1, 2, \ldots \ \textit{and for all } r > -s \\ \frac{d}{dt_s} u &= K_s \equiv B_r \gamma_{-r+s+N}, \ \ s = 0, 1, 2, \ldots \ \textit{and for all } r \leq s + N \end{aligned} \tag{25}$$

where $u = (u_0, \ldots, u_{N-1})^T$, $u_i = u_i(x,t)$, B_r *are Hamiltonian operators defined in (12) with* $\varepsilon_0 = \varepsilon_N = 0$, $u_N = -1$, *where* γ_s *are one-forms given by (18)*

$$\gamma_s = \left(R^\dagger\right)^s \gamma_0, \qquad \gamma_{-s} = \left(\left(R^{-1}\right)^\dagger\right)^{s-1} \gamma_{-1}, \quad s = 1,2,\ldots \tag{26}$$

with γ_0 *and* γ_{-1} *given by*

$$\gamma_0 = (0,\ldots,0,2)^T, \qquad \gamma_{-1} = (u_0^{-1/2}, 0, \ldots, 0)^T. \tag{27}$$

Note that γ_0 and γ_{-1} defined by (27) do satisfy (15) so that $\gamma_0 \in$Ker$\left((R^\dagger)^{-1}\right)$ while $\gamma_{-1} \in\in$Ker$(R^\dagger)$. Explicitly, for the invertible cKdV hierarchy we have

$$\begin{aligned}
J_0 &= \frac{1}{2}\left(u_0\partial + \partial u_0\right), \\
J_i &= \frac{1}{4}\varepsilon_i \partial^3 + \frac{1}{2}\left(u_i\partial + \partial u_i\right), \quad i = 1,\ldots,N-1, \\
J_N &= -\partial.
\end{aligned}$$

Note also that (25) implies

$$B_r\gamma_{-s} = K_{r-s}, \text{ for all } r < s, \qquad B_r\gamma_s = K_{s+r-N} \text{ for all } r \geq N-s \tag{28}$$

and also

$$K_s = R^s K_0, \; s = 1,2,\ldots \text{ with } K_0 = B_0\gamma_0, \tag{29}$$

$$K_{-s} = \left(R^{-1}\right)^{s-1} K_{-1}, \; s = 1,2,\ldots \text{ with } K_{-1} = B_0\gamma_{-1}, \tag{30}$$

so that by the definition above all the vector fields K_s have infinitely many equivalent Hamiltonian representations. If we denote by $(K_s)_i$ the i-th component of the vector field K_s then the first few members of this double-infinite hierarchy are

$$\begin{aligned}
(K_{-1})_i &= -\frac{1}{4}\varepsilon_i\left(u_0^{-1/2}\right)_{xxx} - u_i\left(u_0^{-1/2}\right)_x - \frac{1}{2}u_{ix}u_0^{-1/2} \\
(K_0)_i &= u_{i-1,x} \\
(K_1)_i &= u_{i-2,x} + \frac{1}{4}\varepsilon_{i-1}u_{N-1,xxx} + u_{N-1,x}u_{i-1} + \frac{1}{2}u_{N-1}u_{i-1,x}
\end{aligned}$$

while the first few one-forms γ_s are given by

$$\begin{aligned}
\gamma_{-2} &= \left(\frac{1}{4}\varepsilon_1 u_0^{-1/2}\left(u_0^{-1/2}\left(u_0^{-1/2}\right)_{xx} - \frac{1}{2}\left(u_0^{-1/2}\right)_x^2\right) + \frac{1}{2}u_0^{-3/2}, u_0^{-1/2}, 0, \ldots, 0\right)^T \\
\gamma_{-1} &= (u_0^{-1/2}, 0, \ldots, 0)^T \\
\gamma_0 &= (0,\ldots,0,2)^T \\
\gamma_1 &= (0,\ldots,0,2,u_{N-1})^T \\
\gamma_2 &= \left(0,\ldots,0,2,u_{N-1},u_{N-2} + \frac{1}{4}\varepsilon_{N-1}u_{N-1,xx} + \frac{3}{4}u_{N-1}^2\right)^T.
\end{aligned}$$

Moreover, due to the hereditary property of R, we have that $[K_i, K_j] = 0$ for all $i,j \in \mathbf{Z}$, $d\gamma_i = 0$ (γ_i are all exact one-forms), $L_{K_i}R = 0$ for all $i \in \mathbf{Z}$, $L_{K_i}\gamma_j = 0$ for all $i,j \in \mathbf{Z}$.

Now, it is possible to show that

$$\gamma_s = (P_{s-N+1}, \ldots, P_s)^T, \quad s = 1,2,\ldots \text{ with } P_\alpha = 0 \text{ for } \alpha < 0$$

with P_s defined in (6) being exactly the same as those originally given in the papers of Antonowicz and Fordy. That means that our hierarchy is indeed a negative extension of the cKdV hierarchy considered in [4]. Note also that this hierarchy depends on $N-1$ free parameters $\varepsilon_1, \ldots, \varepsilon_{N-1}$ (since both ε_0 and ε_N are zero, contrary to the case considered in [4] where it depended on N parameters $\varepsilon_1, \ldots, \varepsilon_N$). Let us also remark that in the one-field case (i.e., when $N = 1$ so that only u_0 evolves) our hierarchy becomes equivalent to the dispersionless KdV hierarchy.

Let us now investigate the nature of nonlocalities arising from the invertible cKdV hierarchy. We start by observing that if we split the operators R and R^{-1} into their positive (purely local) and negative (purely nonlocal) parts then we obtain

$$
\begin{aligned}
R &= R_+ + R_- = R_+ + \frac{1}{4}K_0\partial^{-1}\gamma_0^T \\
R^{-1} &= \left(R^{-1}\right)_+ + \left(R^{-1}\right)_- = \left(R^{-1}\right)_+ + K_{-1}\partial^{-1}\gamma_{-1}^T
\end{aligned}
\tag{31}
$$

($R_+ = R_{>-1}$ and $R_- = R_{-1}$ in the notation from the previous chapter) so that both R and R^{-1} have nonlocalities of order 1.

Theorem 7 *The vector fields K_s are local for all $s \in \mathbf{Z}$.*

Proof. We will prove this statement inductively with respect to s and separately for the positive and for the negative part of the hierarchy. First consider the positive hierarchy. The vector K_0 is local. Assume now that K_s is local. Then, due to (31)

$$
K_{s+1} = R(K_s) = R_+(K_s) + \frac{1}{4}K_0\partial^{-1}\gamma_0^T K_s
$$

and obviously $R_+(K_s)$ is local due to the assumption that K_s is local. Since all K_s are symmetries for all γ_s we have that $L_{K_s}\gamma_0 = 0$ which since $d\gamma_0 = 0$ yields $d\left\langle \gamma_0, K_s\right\rangle = 0$ where d is the operator of exterior differentiation and where $\langle\cdot,\cdot\rangle$ is the dual map between cotangent and tangent spaces. Thus, $\gamma_0^T K_s$ is a total derivative so that $\partial^{-1}\gamma_0^T K_s$ is purely local. This completes the inductive step. The proof for the negative part is similar. Since K_{-1} is local we have

$$
K_{s-1} = R^{-1}(K_s) = \left(R^{-1}\right)_+(K_s) + K_{-1}\partial^{-1}\gamma_{-1}^T K_s
$$

and $\partial^{-1}\gamma_{-1}^T K_s$ is purely local by the same argument as above, since $L_{K_s}\gamma_{-1} = 0$. ∎

A similar theorem is valid for one-forms γ_s.

Theorem 8 *The one-forms γ_s are local for all $s \in \mathbf{Z}$.*

Proof. The proof is analogous to the proof of previous theorem: one proves it by induction with respect to s separately for the positive and for the negative hierarchy. We give the proof only for the positive hierarchy. The (nontrivial) one-form γ_0 is local. Assume that γ_s is local. Then, since $R^\dagger = \left(R^\dagger\right)_+ + \left(R^\dagger\right)_- = \left(R^\dagger\right)_+ - \frac{1}{4}\gamma_0\partial^{-1}K_0^T$,

$$
\gamma_{s+1} = \left(R^\dagger\right)_+(\gamma_s) - \frac{1}{4}\gamma_0\partial^{-1}K_0^T\gamma_s.
$$

But obviously $\left(R^\dagger\right)_+(\gamma_s)$ is local due to the assumption while, since $L_{K_0}\gamma_s = 0$, $K_0^T\gamma_s$ is a total derivative so that $\partial^{-1}K_0^T\gamma_s$ is purely local. This completes the induction. ∎

The situation is different when we consider the Poisson operators B_s.

Theorem 9 *The operators $B_0, \ldots, B_N$ are local. All the others operator B_s are nonlocal of order* 1 *with the nonlocal terms of the form*

$$
(B_{N+s})_- = -\frac{1}{4}\sum_{j=1}^{s} K_{j-1}\partial^{-1}K_{s-j+1}^T, \quad s \in \mathbf{Z}_+
$$

$$
(B_{-s})_- = -\sum_{j=1}^{s} K_{-j}\partial^{-1}K_{-s+j-1}^T, \quad s \in \mathbf{Z}_+
$$

Proof. Due to the fact that $R_- = \frac{1}{4}K_0\partial^{-1}\gamma_0^T$ we have, according to formula (24) in Theorem 5

$$(R^s)_- = (R^s)_{-1} = \frac{1}{4}\sum_{j=0}^{s-1} R^j(K_0)\partial^{-1}\left[\left(R^\dagger\right)^{s-1-j}\gamma_0\right]^T = \frac{1}{4}\sum_{j=0}^{s-1} K_j\partial^{-1}\gamma_{s-1-j}^T = \tag{32}$$
$$= \frac{1}{4}\sum_{j=1}^{s} K_{j-1}\partial^{-1}\gamma_{s-j}^T$$

for all $s \in \mathbf{Z}_+$. Thus, for all $s \in \mathbf{Z}_+$ and due to the fact that B_0 is local we have $(B_s)_- = (R^s B_0)_- = (R^s)_- B_0$ which by formula (32) above yields

$$(B_s)_- = \frac{1}{4}\sum_{j=1}^{s} K_{j-1}\partial^{-1}\gamma_{s-j}^T B_0 = \frac{1}{4}\sum_{j=1}^{s} K_{j-1}\partial^{-1}\left(B_0^\dagger\gamma_{s-j}\right)^T = -\frac{1}{4}\sum_{j=1}^{s} K_{j-1}\partial^{-1}\left(B_0\gamma_{s-j}\right)^T$$

since $B_0^\dagger = -B_0$ and due to the fact that $\partial^{-1}\gamma^T B = \partial^{-1}(B^\dagger\gamma)^T$ for any linear pseudo-differential operator B and any column matrix γ with entries depending on x. Thus, since $B_0\gamma_{s-j} = 0$ for $s = 1, \ldots, N$ and $j = 1, \ldots, s$ (γ_{s-j} are the Casimir forms for B_0 for all $s = 1, \ldots, N$ and $j = 1, \ldots, s$ due to Lemma 2) we see that $(B_s)_- = 0$ for $s = 0, \ldots, N$. Moreover, due to (28), $B_0\gamma_{N+s-j} = K_{s-j}$ for $j = 1, \ldots, s$ and thus

$$(B_{N+s})_- = -\frac{1}{4}\sum_{j=1}^{N+s} K_{j-1}\partial^{-1}\left(B_0\gamma_{N+s-j}\right)^T = -\frac{1}{4}\sum_{j=1}^{s} K_{j-1}\partial^{-1}K_{s-j+1}^T.$$

Similarly, due to the fact that $\left(R^{-1}\right)_- = K_{-1}\partial^{-1}\gamma_{-1}^T$ we obtain, by the same formula (24) in Theorem 5 that for all $s \in \mathbf{Z}_+$

$$\left(R^{-s}\right)_- = \left(R^{-s}\right)_{-1} = \sum_{j=0}^{s-1} R^{-j}(K_{-1})\partial^{-1}\left[\left(\left(R^{-1}\right)\dagger\right)^{s-1-j}(\gamma_{-1})\right]^T = \sum_{j=0}^{s-1} K_{-j-1}\partial^{-1}\gamma_{-s+j}^T = \tag{33}$$
$$= \sum_{j=1}^{s} K_{-j}\partial^{-1}\gamma_{-s+j-1}^T$$

and thus by computations similar to the above we obtain

$$(B_{-s})_- = \left(R^{-s}\right)_- B_0 = \sum_{j=1}^{s} K_{-j}\partial^{-1}\gamma_{-s+j-1}^T B_0 = \sum_{j=1}^{s} K_{-j}\partial^{-1}\left(B_0^\dagger\gamma_{-s+j-1}\right)^T =$$
$$= -\sum_{j=1}^{s} K_{-j}\partial^{-1}\left(B_0\gamma_{-s+j-1}\right)^T = -\sum_{j=1}^{s} K_{-j}\partial^{-1}K_{-s+j-1}^T.$$

Thus, all B_s except $B_0, \ldots, B_N$ are nonlocal of order 1. ■

Finally, let us discuss the nonlocalities of symplectic forms Ω_s defined through (14). In order to establish the leading nonlocal term of Ω_s by applying Theorems 3 and 5 to $\Omega_s = \left(R^\dagger\right)^s\Omega_0$ we need first to establish the leading nonlocal term of Ω_0.

Theorem 10 *Ω_0 has nonlocality of first order and its leading nonlocal term is given by*

$$(\Omega_0)_- = \frac{1}{4}\sum_{j=0}^{N-1}\gamma_j\partial^{-1}\gamma_{N-1-j}^T. \tag{34}$$

One can prove this formula by showing that (34) is a simultaneous solution to all equations

$$(B_s\Omega_0)_- = (R^s)_-, \quad s \in \mathbf{Z}.$$

We skip the proof as the computations involved are similar to those for nonlocal parts of B_s operators.

Now we can calculate the nonlocalities of Ω_s by applying Theorems 3 and 5 to $\Omega_s = \left(R^\dagger\right)^s\Omega_0$. The result of this calculation is presented (without proof) in the theorem below.

Theorem 11 *The closed two-forms $\Omega_s = \left(R^\dagger\right)^s \Omega_0$ are nonlocal of first order (weakly nonlocal) with nonlocal terms given by*

$$\left(\Omega_s\right)_- = \frac{1}{4}\sum_{j=0}^{N+s-1}\gamma_j\partial^{-1}\gamma_{s+N-1-j}^T, \quad s \in \mathbf{Z}_+,$$
$$\left(\Omega_{-s}\right)_- = \frac{1}{4}\sum_{j=0}^{N-s-1}\gamma_j\partial^{-1}\gamma_{-s+N-1-j}^T + \sum_{j=1}^{s}\gamma_{-j}\partial^{-1}\gamma_{-s+j-1}^T, \quad s \in \mathbf{Z}_+.$$

5 Invertible coupled Harry Dym hierarchy

There is a second possibility for choosing the constants in J_0 and J_N so that both operators become invertible, which guarantees the existence of both positive and negative hierarchies, namely, to take, as before $\varepsilon_N = 0$ and then to put $u_0 = 0$. The operators J_i attain then the form

$$J_0 = \frac{1}{4}\varepsilon_0\partial^3,$$
$$J_i = \frac{1}{4}\varepsilon_i\partial^3 + \frac{1}{2}\left(u_i\partial + \partial u_i\right), \quad i = 1, \ldots, N-1,$$
$$J_N = \frac{1}{2}(u_N\partial + \partial u_N).$$

We can therefore define the invertible coupled Harry Dym (invertible cHD) hierarchy as follows:

Definition 12 *The invertible N-component coupled Harry Dym hierarchy is the family of flows*

$$\frac{d}{dt_{-s}}u = K_{-s} \equiv B_r\gamma_{-r-s}, \quad s = 1, 2, \ldots \text{ and for all } r > -s$$
$$\frac{d}{dt_s}u = K_s \equiv B_r\gamma_{-r+s+N}, \quad s = 0, 1, 2, \ldots \text{ and for all } r \leq s+N \tag{35}$$

where $u = (u_1, \ldots, u_N)^T$, $u_i = u_i(x,t)$, B_r are hamiltonian operators defined in (12) with $\varepsilon_N = 0$, $u_0 = 0$ and where γ_s are one-forms given by (18)

$$\gamma_s = \left(R^\dagger\right)^s\gamma_0, \qquad \gamma_{-s} = \left(\left(R^{-1}\right)^\dagger\right)^{s-1}\gamma_{-1}, \quad s = 1, 2, \ldots \tag{36}$$

with

$$\gamma_0 = \left(0, \ldots, 0, u_N^{-1/2}\right)^T \in Ker\left(\left(R^\dagger\right)^{-1}\right), \qquad \gamma_{-1} = (-2, 0, \ldots, 0)^T \in Ker(R^\dagger) \tag{37}$$

(again γ_0 and γ_{-1} are chosen as in (15)). Thus, formally, both hierarchies have the same algebraic structure (apart from the fact that now the nontrivial variables are $u_1, \ldots, u_N$); the formulas (28)-(30) are still valid, but of course the exact form of invariant one-forms γ_i and vector fields K_i differ. Explicitly, for the invertible cHD hierarchy we have

$$\left(K_{-2}\right)_i = u_{i+1,x} - \frac{1}{\varepsilon_0}\varepsilon_i u_{1,x} - \frac{4}{\varepsilon_0}u_i\partial^{-1}u_1 - \frac{2}{\varepsilon_0}u_{i,x}\partial^{-2}u_1$$
$$\left(K_{-1}\right)_i = u_{i,x}$$
$$\left(K_0\right)_i = \frac{1}{4}\varepsilon_{i-1}\left(u_N^{-1/2}\right)_{xxx} + u_{i-1}\left(u_N^{-1/2}\right)_x + \frac{1}{2}u_{i-1,x}u_N^{-1/2}$$

(with $u_i = 0$ for $i < 1$ or $i > N$) so that the negative part is no longer local. Below we prove that the positive part is still local. Note also that this hierarchy depends now on N parameters: $\varepsilon_0, \ldots, \varepsilon_{N-1}$. The

first few conserved one-forms of the invertible cHD hierarchy are

$$\begin{aligned}
\gamma_{-2} &= \left(\frac{4}{\varepsilon_0}\partial^{-2}u_1, -2, 0, \ldots, 0\right)^T \\
\gamma_{-1} &= (-2, 0, \ldots, 0)^T \\
\gamma_0 &= \left(0, \ldots, 0, u_N^{-1/2}\right)^T \\
\gamma_1 &= \left(0, \ldots, 0, u_N^{-1/2}, -\frac{1}{4}\varepsilon_{N-1}\left[u_N^{-1}\left(u_N^{-1/2}\right)_{xx} - \frac{1}{8}u_N^{-7/2}\left(u_{N,x}\right)^2\right] - \frac{1}{2}u_N^{-3/2}u_{N-1}\right)^T.
\end{aligned}$$

As in the cKdV case, the positive part of the above hierarchy coincides with the coupled Harry Dym hierarchy in [6] after putting $a = 0$. We also see that the negative part of the sequence of γ_i is nonlocal; we prove below that the positive γ_i's are local. Let us thus consider the leading nonlocal parts of all the objects of the invertible cHD hierarchy. We first establish the nonlocal parts of the recursion operator R and its inverse R^{-1}. We obtain (cf. (31))

$$\begin{aligned}
R &= R_+ + R_- = R_+ - K_0\partial^{-1}\gamma_0^T \qquad (38)\\
R^{-1} &= \left(R^{-1}\right)_+ + \left(R^{-1}\right)_- = \left(R^{-1}\right)_+ + \frac{2}{\varepsilon_0}u\partial^{-2}\gamma_{-1}^T + \frac{1}{\varepsilon_0}K_{-1}\partial^{-3}\gamma_{-1}^T.
\end{aligned}$$

Thus, contrary to the cKdV case, R^{-1} has nonlocality of order 3 while R is still nonlocal of order 1. Similarly to the cKdV case, we obtain by induction that

Theorem 13 *The vector fields K_i and the conserved one-forms γ_i are local for all $i \in \mathbf{Z}_+$.*

The structure of nonlocalities in the operators B_s differs from the cKdV case.

Theorem 14 *The operators $B_0, \ldots, B_N$ are local. All the others operator B_s are nonlocal with the leading nonlocal terms of the form*

$$\begin{aligned}
\left(B_{s+N}\right)_{-1} &= \sum_{j=1}^{s} K_{j-1}\partial^{-1}K_{s-j+1}^T, \quad s \in \mathbf{Z}_+ \\
\left(B_{-s}\right)_{-3} &= -\frac{1}{\varepsilon_0}\sum_{j=0}^{s-1} K_{-j-1}\partial^{-3}K_{-s+j}^T, \quad s \in \mathbf{Z}_+
\end{aligned}$$

Again, one proves this theorem by applying Theorem 3 and Theorem 5 to the operator $B_s = R^s B_0$. Thus, the negative operators are nonlocal of order 3. The above theorem can now be used to discuss the nonlocalities of the closed two-forms Ω_s defined through (14). We begin by establishing the form of the nonlocal part of Ω_0.

Theorem 15 *Ω_0 has nonlocality of first order and its leading nonlocal term is given by*

$$\left(\Omega_0\right)_- = -\sum_{j=0}^{N-1} \gamma_j\partial^{-1}\gamma_{N-1-j}^T. \qquad (39)$$

This theorem can be proved just as in the cKdV case. Now, by applying again Theorem 3 and Theorem 5 to the operator $\Omega_s = \left(R^\dagger\right)^s \Omega_0$ with Ω_0 given by (39) we obtain the following theorem.

Theorem 16 *The symplectic operators $\Omega_s = \left(R^\dagger\right)^s \Omega_0$ are nonlocal of first order for $s > 0$ and of third order for $s < 0$ and their leading nonlocal terms are given by*

$$\begin{aligned}
\left(\Omega_s\right)_- &= -\sum_{j=0}^{N+s-1} \gamma_j\partial^{-1}\gamma_{s+N-1-j}^T, \quad s \in \mathbf{Z}_+ \\
\left(\Omega_{-s}\right)_{-3} &= -\frac{1}{\varepsilon_0}\sum_{j=1}^{s} \gamma_{-j}\partial^{-3}\gamma_{-s+j-1}^T, \quad s \in \mathbf{Z}_+
\end{aligned}$$

6 Conclusions

In this paper we have stated some natural conditions under which the coupled Korteweg-de Vries and the coupled Harry Dym hierarchies, introduced in the paper [1] in the two-field case and developed in papers [2]-[7], are invertible, i.e., possess negative parts. We studied the structure of nonlocalities of various tensor invariants of these hierarchies. It turns out that all the vector fields and conserved one-forms of the invertible cKdV hierarchy are local while all its Poisson operators are either local or at most weakly nonlocal. Finally, all symplectic operators of the hierarchy are weakly nonlocal. In the case of the invertible cHD hierarchy only vector fields and conserved one-forms of the positive part are local, while Poisson operators of this hierarchy are either local or have nonloclities of first or third order. Moreover, all symplectic operators of the invertible cHD hierarchy are nonlocal of first or third order. The main tool for our considerations was a generalization of formulae (7) and (8) from [9], that is, Theorems 3 and 5 above.

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